PROOF

The formulas used to derive the complex impedance of an inductor from the measurement of S11 are as follows [1].

$$Z = R + jX = Z_0 \frac{1+\Gamma}{1-\Gamma}\ [\Omega] \quad 0.1$$

Here, $Z_0$is the instrument impedance (50 ohms) and Γ is the reflection coefficient, which coincides with the scattering parameter S11. The Γ parameter, like S11, is a complex number consisting of a real part and an imaginary part.

$$\Gamma = \Gamma_r + j\Gamma_i \quad 0.2$$

By substituting the expression of Γ into the impedance formula, the impedance is obtained as a complex quantity.

$$Z = R + jX = Z_0 \frac{1+\Gamma_r+j\Gamma_i}{1-\Gamma_r-j\Gamma_i}\ [\Omega] \quad 0.3$$

By rationalizing the denominator and expanding the numerator, it is possible to separate the real part R and the imaginary part X of the impedance Z.

$$Z = R + jX = Z_0 \frac{1+\Gamma_r+j\Gamma_i}{1-\Gamma_r-j\Gamma_i} = Z_0 \frac{(1+\Gamma_r+j\Gamma_i)(1-\Gamma_r+j\Gamma_i)}{(1-\Gamma_r-j\Gamma_i)(1-\Gamma_r+j\Gamma_i)} = Z_0 \frac{1-\Gamma_r^2-\Gamma_i^2+j2\Gamma_i}{(1-\Gamma_r)^2+{\Gamma_i}^2}\ [\Omega] \quad 0.4$$

$$R = Z_0 \frac{1-{\Gamma_r}^2-{\Gamma_i}^2}{(1-\Gamma_r)^2+\Gamma_i^2}\ [\Omega] \quad 0.5$$

$$X = Z_0 \frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}\ [\Omega] \quad 0.6$$

Now let us examine the toroid, with reference to Figure 1.

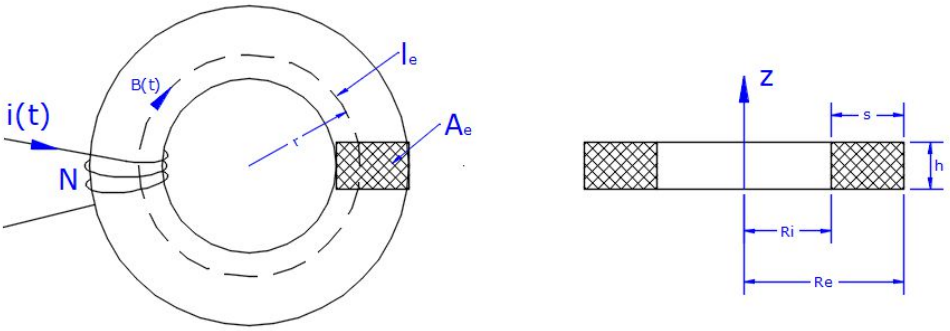

Figure 1: Winding of N turns around a toroid with a rectangular cross section.

We list the main equations governing magnetic induction:

$\oint H \cdot dl = N\ i$ The Ampère's law (where N is the number of turns); 0.7

$B = \mu H$ The relationship between magnetic induction B and magnetic field H; 0.8

$\Phi = B \cdot A_e$ The relationship between magnetic flux and magnetic induction B; 0.9

$L = N\frac{\Phi}{i}$ The definition of inductance. 0.10

To derive the inductance of a winding with N turns on a toroid with a square cross section, we must determine the magnetic flux circulating in the toroid. To do this, we start from Ampère's law 0.7, where the magnetic field vector is replaced by the magnetic flux density vector 0.8.

$$\oint B \cdot dl = N\ \mu\ i \quad 0.11$$

We choose to calculate the circulation of the magnetic flux density vector along the dashed closed path shown in Figure 1 left, since it is convenient. In fact, the scalar product of the vector B along the chosen closed path is constant.

$$\oint B \cdot dl = B\ 2\pi r \quad 0.12$$

$$B = \frac{N\ \mu\ i}{2\pi\ r} \quad 0.13$$

From this, it is clear that the scalar value of B is constant along the closed path, but it depends on the distance from the center of the toroid.

To calculate the magnetic flux, we use the relation between flux and magnetic induction 0.9, substituting the expression 0.13 obtained for B. The effective area A_e becomes a double integral over the rectangular cross section of the toroid.

$$\Phi = \int_0^h dz \int_{Ri}^{Re} \frac{N\,\mu\,i}{2\pi\,r} dr \qquad 0.14$$

The first integral yields the height h of the toroid, while the second integral leads to a logarithmic expression involving the inner R_i and outer radii R_e.

$$\Phi = h \frac{N\,\mu\,i}{2\pi} ln\left(\frac{Re}{Ri}\right) \qquad 0.15$$

Finally, the magnetic flux obtained 0.15 can be substituted into the definition of inductance 0.10.

$$\mathrm{L} = \frac{h\,N^2\,\mu}{2\pi} ln\left(\frac{Re}{Ri}\right) \text{ [H]} \qquad 0.16$$

By expressing the permeability as the product of the vacuum permeability and the relative permeability $\mu = \mu_0 \mu_r$ and using the known value of the vacuum permeability $\mu_0 = 4\pi\ 10^{-7}$ the inductance formula 0.16 is obtained.

$$\mathrm{L} = 2h\,N^2 \mu_r ln\left(\frac{Re}{Ri}\right) 10^{-7} \text{ [H]} \qquad 0.17$$

Where: μ_r is the relative permeability; N is the number of turns; h is the height of the toroid (in meters); R_e is the outer diameter of the toroid (in meters) and R_i is the inner diameter of the toroid (in meters).

If we define L_0 as the inductance for $\mu_r = 1$, the inductance can be written as the product of L_0 and the relative permeability.

$$L_0 = 2h\,N^2 ln\left(\frac{Re}{Ri}\right) 10^{-7} \text{ [H]} \qquad 0.18$$

$$L = L_0 \mu_r \text{ [H]} \qquad 0.19$$

The impedance of a ferrite inductor can be expressed in terms of the complex relative permeability [2].

$$Z = j\omega L_0 (\mu_r' - j\mu_r'') = \omega L_0 \mu_r'' + j\omega L_0 \mu_r' = R + jX\ [\Omega] \qquad 0.20$$

The real μ_r' and imaginary μ_r'' parts of the complex permeability μ_r give rise respectively to the reactance X and the resistance R of the impedance Z.

At low frequencies, around 1 kHz, the relative permeability of a solenoid wound on a ferrite core is considered to lie in the initial magnetization curve. Consequently, the inductance is also considered to be the initial inductance. For this reason, both the relative permeability and the inductance are denoted with the subscript “i”, which stands for *initial*.

Therefore, the initial inductance, measured with an LCR meter at 1 kHz, also follows the same inductance relation 0.19, from which L_0 can be derived.

$$L_0 = \frac{L_i}{\mu_i} \text{ [H]} \qquad 0.21$$

Finally, from the impedance formula 0.20, the parameters $\mu'$and $\mu''$are obtained by substituting the appropriate expression 0.21 for L_0. The resulting equations are those used in the Excel spreadsheet.

$$\mu_r' = \frac{X}{2\pi f \cdot L_0} = \frac{X \cdot \mu_i}{2\pi f \cdot L_i} \qquad 0.22$$

$$\mu_r'' = \frac{R}{2\pi f \cdot L_0} = \frac{R \cdot \mu_i}{2\pi f \cdot L_i}\,. \qquad 0.23$$

REFERENCES

[1] D. M. Pozar: Microwave Engineering, 3° ed. John Wiley & Sons, 2005.

[2] E. C. Snelling: Soft Ferrites, ed. Ilieffe Books Ltd, 1969.